Exercise 5

1. Determine whether the following parametrized surfaces are regular in D:

   (a) $\mathbf{x}(u,v) = (u, v, uv)$, $(u,v) \in D = (0,1) \times (0,1)$

   (b) $\mathbf{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v)$, $(u,v) \in D = (0, 2\pi) \times \mathbb{R}$

   (c) $\mathbf{x}(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$, $(u,v) \in D = \mathbb{R}^2$

2. Let $D \subset \mathbb{R}^2$ be a connected domain. Let $f : D \to \mathbb{R}$ be a smooth function and $\mathbf{x} : D \to \mathbb{R}^3$ be a smooth map defined by

$$\mathbf{x}(u,v) = (u, v, f(u,v))$$

   (**Note.** This surface is called the **Graphical surfaces**).

   (a) Show that the surface $\mathbf{x}(D)$ is a regular parametrized surface.

   (b) Hence, determine whether the following surfaces are regular:

   (i) $S = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 - y^2\}$

   (ii) $S = \{(x,y,z) \in \mathbb{R}^3 : z = x - y + 1\}$

   (c) Prove that the tangent plane of the graphical surface at the point $p = \mathbf{x}(u,v)$ is

$$z = f(u,v) + f_u(x-u) + f_v(y-v)$$

   (**Note.** f_u and f_v denotes the $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ respectively at (u,v).)

   (d) Determine the tangent plane of the surface $S = \{(x,y,z) \in \mathbb{R}^3 : z = x^2 + y^2\}$ at the point $(4, -3, 25) \in S$.

3. If all normal lines to a connected surface S passes through a fixed point $\mathbf{p} \in \mathbb{R}^3$, show that S is contained in a sphere.

4. Compute the first fundamental form of the following regular parametrized surfaces of revolution:

   (a) $\mathbf{x}(u,v) = (a \cosh u \cos v, a \cosh u \sin v, c \sinh u)$ (hyperboloid of revolution of one sheet);

   (b) $\mathbf{y}(u,v) = (a \sinh u \cos v, a \sinh u \sin v, c \cosh u)$ (hyperboloid of revolution of two sheets).

5. Let $\mathbf{x}(u,v)$ be a regular parametrized surface S and $\mathbf{r}(t) = \mathbf{x}(u(t), v(t))$ be a curve lying on the surface S, for $a < t < b$.

   (a) Using chain rule, show that $\mathbf{r}'(t) = \mathbf{x}_u \dfrac{du}{dt} + \mathbf{x}_v \dfrac{dv}{dt}$.

   (b) Hence, show that the arc length of $\mathbf{r}(t)$ is given by

$$\ell = \int_a^b \sqrt{\begin{pmatrix} \frac{du}{dt} & \frac{dv}{dt} \end{pmatrix} I \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix}}\, dt$$

   where I is the first fundamental form of $\mathbf{x}(u,v)$.

6. For the following regular parametrized surface, find the first fundamental form and the surface area under the bounded domain D.

   (a) $\mathbf{x}(u,\theta) = (u^3 \cos\theta, u^3 \sin\theta, u)$, $(u,\theta) \in D = (0,1) \times (0, 2\pi)$.

   (b) $\mathbf{x}(u,v) = (u \cos v, u \sin v, v)$, $(u,v) \in D = (-1,1) \times [0, 2\pi)$.

   (c) $\mathbf{x}(u,v) = (2(1 + \cos v)\cos u, 2(1 + \cos v)\sin u, 2\sin v)$, $(u,v) \in D = (0,1) \times (0, 2\pi)$.
   (**Note.** This parametrized surface is called the **Horn Torus**.)